

# DEPENDENCE OF SUPERTROPICAL EIGENSPACES

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**ABSTRACT.** We study the pathology that causes tropical eigenspaces of distinct supertropical eigenvalues of a nonsingular matrix  $A$ , to be dependent. We show that in lower dimensions the eigenvectors of distinct eigenvalues are independent, as desired. The index set that differentiates between subsequent essential monomials of the characteristic polynomial, yields an eigenvalue  $\lambda$ , and corresponds to the columns of the eigenmatrix  $A + \lambda I$  from which the eigenvectors are taken. We ascertain the cause for failure in higher dimensions, and prove that independence of the eigenvectors is recovered in case a certain “difference criterion” holds, defined in terms of disjoint differences between index sets of subsequent coefficients. We conclude by considering the eigenvectors of the matrix  $A^\nabla := \frac{1}{\det(A)} \text{adj}(A)$  and the connection of the independence question to generalized eigenvectors.

## 1. INTRODUCTION

Although supertropical matrix algebra as developed in [20, 21] follows the general lines of classical linear algebra (i.e., a Cayley-Hamilton Theorem, correspondence between the roots of the characteristic polynomial and eigenvalues, Kramer’s rule, etc.), one encounters the anomaly in [21, Remark 5.3 and Theorem 5.6] of a matrix whose supertropical eigenvalues are distinct but whose corresponding supertropical eigenspaces are dependent. In this paper we examine how this happens, and give a criterion for the supertropical eigenspaces to be dependent, which we call the **difference criterion**, cf. Definition 3.1 and Theorem 3.4. A pathological example (3.3) is studied in depth to show why the difference criterion is critical. We resolve the difficulty in general in Theorem 3.11 by passing to powers of  $A$  and considering generalized supertropical eigenspaces.

**1.1. The tropical algebra and related structures.** We start by discussing briefly the max-plus algebra, its refinements, and their relevance to applications.

The use of the max-plus algebra in tropical mathematics was inspired by the function  $\log_t$ , as the base  $t$  of the logarithm approaches 0. In the literature, this structure is usually studied via valuations (see [16] and [17]) over the field  $K = \mathbb{C}\{\{t\}\}$  of Puiseux series with powers in  $\mathbb{Q}$  (resp.  $\mathbb{R}$ , to the ordered group  $(\mathbb{Q}, +, \geq)$  (resp.  $(\mathbb{R}, +, \geq)$ ). The valuation is given by the lowest exponent appearing nontrivially in the series (indeed  $v(ab) = v(a) + v(b)$  and  $v(a + b) \geq \min(v(a), v(b))$ ). Then, we look at the dual structure obtained by defining  $\text{trop}(a) = -\text{val}(a)$  and denoted as the tropicalization of  $a \in K$ . By setting  $\text{trop}(a + b)$  to be  $\max\{\text{trop}(a), \text{trop}(b)\}$ , it is obvious that the tropical structure deals with the uncertainty of equality in the valuation, in the form of  $\text{trop}(a + a) = \text{trop}(a)$  (also equals to  $\text{trop}(-a)$ ).

**1.2. The max-plus algebra.** The **tropical max-plus semifield** is an ordered group  $\mathcal{T}$  (usually the additive group of real numbers  $\mathbb{R}$  or the set of rational numbers  $\mathbb{Q}$ ), together with a formal element  $-\infty$  adjoined. The ordered group  $\mathcal{T}$  is made into a semiring equipped with the operations

$$a \oplus b = \max\{a, b\} \quad \text{and} \quad a \odot b = a + b,$$

denoted here as  $a + b$  and  $ab$  respectively (see [1], [14] and [15]). The unit element  $1_{\mathcal{T}}$  is really the element  $0 \in \mathbb{Q}$ , and  $-\infty$  serves as the zero element.

Tropicalization enables one to simplify non-linear questions by putting them into a linear setting (see [13]), which can be applied to discrete mathematics (see [4]), optimization (see [10]) and algebraic geometry (see [14]).

In [12] Gaubert and Sharify introduce a general scaling technique, based on tropical algebra, which applies in particular to the companion form, determining the eigenvalues of a matrix polynomial. Akian, Gaubert and Guterman show in [3] that several decision problems originating from max-plus or tropical convexity are equivalent to zero-sum two player game problems.

[25] is a collection of papers put together by Litvinov and Sergeev. One main theme is the Maslov dequantization applied to traditional mathematics over fields, built on the foundations of idempotent analysis, tropical algebra, and tropical geometry. Applications of idempotent mathematics were introduced by Litvinov and Maslov in [24].

On the side of pure mathematics, contributions are made in [25] to idempotent analysis, tropical algebras, tropical linear algebra and tropical convex geometry. Elaborate geometric background with applications to problems in classical (real and complex) geometry can be found in [26]. Here Mikhalkin viewed the tropical structure as a branch of geometry manipulating with certain piecewise-linear objects that take over the role of classical algebraic varieties and describes hypersurfaces, varieties, morphisms and moduli spaces in this setting.

Extensive mathematical applications have been made in combinatorics. In this max-plus language, we may use notions of linear algebra to interpret combinatorial problems. In [23] Jonczyk presents some problems described by the Path algebra and solved by means of min and max operations. Combinatorial overviews are given in [7], [8] of Butkovic and [9] of Butkovic and Murfitt, which focus on presenting a number of links between basic max-algebraic problems on the one hand and combinatorial problems on the other hand. This indicates that the max-algebra may be regarded as a linear-algebraic encoding of a class of combinatorial problems.

**1.3. Supertropical algebra.** We pass to the **supertropical semiring**, equipped with the ghost ideal  $\mathcal{G} := \mathcal{T}^\nu$ , as established and studied by Izhakian and Rowen in [18] and [19].

We denote as  $R = \mathcal{T} \cup \mathcal{G} \cup \{-\infty\}$  the “standard” supertropical semiring, which contains the so-called tangible elements of the structure and where we have a projection  $R \rightarrow \mathcal{G}$  given by  $a \mapsto a^\nu$  for  $a \in \mathcal{T}$  (and which is the identity map on  $\mathcal{G}$ ).  $\{a^\nu \in \mathcal{G}, \forall a \in \mathcal{T}\}$  are the ghost elements of the structure, as defined in [19]. We write  $0_R$  for  $-\infty$ , to stress its role as the zero element. On the one hand,  $\mathcal{G}$  is a copy of the max-plus semifield, so  $R$  can be viewed as a cover of the max-plus semifield.

The supertropical semiring enables us to distinguish between a maximal element  $a$  that is attained only once in a sum, i.e.,  $a \in \mathcal{T}$  which is invertible, and a maximum that is being attained at least twice, i.e.,  $a + a = a^\nu \in \mathcal{G}$ , which is not invertible. We do not distinguish between  $a + a$  and  $a + a + a$  in this structure. Note that  $\nu$  projects the standard supertropical semiring onto  $\mathcal{G}$ , which can be identified with the usual tropical structure.

In this new supertropical sense, we use the following order relation to describe two elements that are equal up to a ghost supplement:

**Definition 1.1.** Let  $a, b$  be any two elements in  $R$ . We say that  $a$  **ghost surpasses**  $b$ , denoted  $a \models_{gs} b$ , if  $a = b + ghost$ . That is,  $a = b$  or  $a \in \mathcal{G}$  with  $a^\nu \geq b^\nu$ .

We say  $a$  is  $\nu$ -equivalent to  $b$ , denoted by  $a \cong_\nu b$ , if  $a^\nu = b^\nu$ . That is, in the tropical structure,  $\nu$ -equivalence projects to equality.

**Important properties of  $\models_{gs}$ :**

- (1)  $\models_{gs}$  is a partial order relation (see [21, Lemma 1.5]).
- (2) If  $a \models_{gs} b$  then  $ac \models_{gs} bc$ .
- (3) If  $a \models_{gs} b$  and  $c \models_{gs} d$  then  $a + c \models_{gs} b + d$  and  $ac \models_{gs} bd$ .
- (4) If  $a \models_{gs} b$  and  $a \in \mathcal{T}$ , then  $a = b$ .

Considering this relation, we regain basic algebraic properties that were not accessible in the usual tropical setting, such as multiplicativity of the tropical determinant, the near multiplicativity of the tropical adjoint, the role of roots in the factorization of polynomials, the role of the determinant in matrix singularity, a matrix that acts like an inverse, common behavior of similar matrices, classical properties of  $\text{adj}(A)$ , and the use of elementary matrices. Tropical eigenspaces and their dependences are of considerable interest, as one can see in [2], [5], [7], [18], [21] and [29].

Many of these properties will be formulated in the Preliminaries section. We would also like to attain a supertropical analog to the classical eigenspace decomposition, (i.e., eigenvectors corresponding to distinct eigenvalues are linearly independent, and the generalized eigenvectors generate  $R^n$ ), but we encounter the example of [21, Example 5.7] where the eigenvectors of distinct eigenvalues are supertropically dependent, extensively studied in Section 3.2. Our objective in this paper is to understand how such an example arises, and how it can be circumvented, either by introducing the **difference criterion** of Definition 3.1 or by passing to generalized eigenspaces in § 3.3.3.

## 2. PRELIMINARIES

In this section, we present well-known and recent results of tropical polynomials. Then we introduce properties of matrices and vectors in the tropical structure, with definitions extended to the supertropical framework.

### 2.1. Tropical Polynomials.

*Notation 2.1.*

Throughout, for each element  $a \in R$ , we choose an element  $\hat{a} \in \mathcal{T}$  such that  $\hat{a} \cong_\nu a$ . (We define  $0_R^\nu = 0_R$ , so  $\widehat{0_R^\nu} = 0_R$ .)

Likewise, for  $\mathbf{a} = (a_1, \dots, a_n)$ ,  $\mathbf{a}^\nu$  denotes  $(a_1^\nu, \dots, a_n^\nu)$  and  $\widehat{\mathbf{a}}$  denotes  $(\hat{a}_1, \dots, \hat{a}_n)$ . The same holds for matrices and for polynomials (according to their coefficients).

**Definition 2.2.** Let  $k \in \mathbb{N}$ . Defining  $b = a^k$  to be the tropical product of  $a$  by itself  $k$  times (i.e.,  $a^{\odot k} = a \odot \cdots \odot a = a + \cdots + a = ka$ ), we may consider that  $a$  is a  $k$ -**root** of  $b$ , denoted as  $a = \sqrt[k]{b}$ . This operation is well-defined on  $\mathcal{T}$ .

Clearly, any tropical polynomial takes the value of the dominant monomial along the  $\mathcal{T}$ -axis. That having been said, it is possible that some monomials in the polynomial would not dominate for any  $x \in \mathcal{T}$ .

**Definition 2.3.** Let  $f(x) = \sum_{i=0}^n \alpha_i x^{n-i} \in R[x]$  be a tropical polynomial. We call monomials in  $f(x)$  that dominate for some  $x \in R$  **essential**, and monomials in  $f(x)$  that do not dominate for any  $x \in R$  **inessential**. We write  $f^{es}(x) = \sum_{k \in I} \alpha_k x^{n-k} \in R[x]$ , where  $\alpha_k x^{n-k}$  is an essential monomial  $\forall k \in I$ , called **the essential polynomial** of  $f$ .

In the classical sense, a root of a tropical polynomial can only be  $0_R$ , which occurs if and only if the polynomial has constant term  $0_R$ . We would like the roots to indicate the factorization of the polynomial, which leads to the following tropical definition of a root.

**Definition 2.4.** We define an element  $r \in R$  to be a **root** of a tropical polynomial  $f(x)$  if  $f(r) \models_{gs} 0_R$ , i.e.,  $f(r)$  is a ghost.

We refer to roots of a polynomial being obtained as a simultaneous value of two leading tangible monomials as **corner roots**, and to roots that are being obtained from one leading ghost monomial as **non-corner roots**. We factor polynomials viewing them as functions. Then, for every corner root  $r$  of  $f$ , we may write  $f$  as  $(x + r)^k g(x)$  for some  $g(x) \in R[x]$  and  $k \in \mathbb{N}$ , where  $k$  is the difference between the exponents of the tangible essential monomials attaining  $r$ .

**2.2. Matrices.** As defined over a ring, for matrices  $A = (a_{i,j}) \in M_{n \times m}(R)$ ,  $B = (b_{i,j}) \in M_{s \times t}(R)$

$$\begin{cases} A + B = (c_{i,j}) : c_{i,j} = a_{i,j} + b_{i,j}, & \text{defined iff } n = s, m = t, \\ AB = (d_{i,j}) : d_{i,j} = \sum_{k \in [n]} a_{i,k} b_{k,j}, & \text{defined iff } m = s. \end{cases}$$

**Definition 2.5.** Let  $\pi \in S_n$  and  $A = (a_{i,j}) \in M_n(R)$ . The **permutation  $\pi$  of  $A$**  is the word

$$a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)}.$$

The word  $a_{1,1} a_{2,2} \cdots a_{n,n}$  is denoted as the **identity or Id-permutation**, corresponding to the diagonal of  $A$ . We write a permutation of  $A$  as a product of disjoint cycles  $C_1, \dots, C_t$ , where  $\{C_i\}$  corresponds to the disjoint cycles composing  $\pi$ .

We define the tropical **trace** and **determinant** of  $A$  to be

$$\text{tr}(A) = \sum_{k \in [n]} a_{k,k} \quad \text{and} \quad \det(A) = \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)},$$

respectively.

In the special case where  $A \in M_n(R)$ , we refer to any entry attaining the trace as a **dominant diagonal entry**. We call  $a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$  the **weight** contributed by  $\sigma$  to the determinant, and any permutation whose weight has the same  $\nu$ -value as the determinant is a **dominant permutation of  $A$** .

If there is a single dominant permutation, its weight equals the determinant.

Unlike over a field, the tropical concepts of singularity, invertability and factorizability do not coincide. We would like the determinant to indicate the singularity of a matrix. Hence, we define a matrix  $A \in M_n(R)$  to be **tropically singular** if there exist at least two different dominant permutations. Otherwise the matrix is **tropically nonsingular**. Consequently, a matrix  $A \in M_n(R)$  is supertropically singular if  $\det(A) \models_{gs} 0_R$  and supertropically nonsingular if  $\det(A) \in \mathcal{T}$ . A matrix  $A$  is **strictly singular** if  $\det(A) = 0_R$ .

A surprising result in this context is that the product of two nonsingular matrices might be singular, but we do have:

**Theorem 2.6.** *For  $n \times n$  matrices  $A, B$  over the supertropical semiring  $R$ , we have*

$$\det(AB) \models_{gs} \det(A) \det(B).$$

This theorem has been proved in [20, Theorem 3.5] due to considerations of graph theory, but also in [11, Proposition 2.1.7] by using the transfer principles (see [2, Theorem 3.3 and Theorem 3.4]). These theorems allow one to obtain such results automatically in a wider class of semirings, including the supertropical semiring.

**Definition 2.7.** Suppose  $\mathcal{R}$  is a semiring. An  $\mathcal{R}$ -**module**  $V$  is a semigroup  $(V, +, 0_V)$  together with scalar multiplication  $\mathcal{R} \times V \rightarrow V$  satisfying the following properties for all  $r_i \in \mathcal{R}$  and  $v, w \in V$ :

- (1)  $r(v + w) = rv + rw$
- (2)  $(r_1 + r_2)v = r_1v + r_2v$
- (3)  $(r_1r_2)v = r_1(r_2v)$
- (4)  $1_{\mathcal{R}}v = v$
- (5)  $r \cdot 0_V = 0_V$
- (6)  $0_{\mathcal{R}} \cdot v = 0_V$ .

For any semiring  $R$ , let  $R^n$  be the free module of rank  $n$  over  $R$ . We define the **standard base** to be  $e_1, \dots, e_n$ , where

$$e_i = \begin{cases} 1_{\mathcal{T}} = 1_R, & \text{in the } i^{th} \text{ coordinate} \\ 0_{\mathcal{T}} = 0_R, & \text{otherwise} \end{cases}.$$

The tropical **identity matrix** is the  $n \times n$  matrix with the standard base for its columns. We denote this matrix as  $I_{\mathcal{T}} = I$ .

A matrix  $A \in M_n(R)$  is **invertible** if there exists a matrix  $B \in M_n(R)$  such that  $AB = BA = I$ .

From now on  $\mathcal{F} := \mathcal{T} \cup \mathcal{G} \cup \{0_{\mathcal{F}}\}$ , where its set  $\mathcal{T}$  is presumed to be a group, and  $\mathcal{G}$  is its ghost elements. We write  $V = \mathcal{F}^n$ , with the standard base  $\{e_1, \dots, e_n\}$ .

**Definition 2.8.** We define vectors  $v_1, \dots, v_k$  in  $V$  to be **(supertropically) dependent** if there exist  $a_1, \dots, a_k \in \mathcal{T}$  such that  $\sum_{i \in [k]} a_i v_i \models_{gs} \vec{0}_{\mathcal{F}}$ . Otherwise, this set of tropical vectors is called **independent**.

We say that subspaces  $V_1, \dots, V_k$  of  $\mathcal{F}^n$ , are **(supertropically) dependent**, if there are tangible  $v_i \in V_i$  which are (supertropically) dependent.

By [20, Theorem 6.5], vectors  $v_1, \dots, v_n \in \mathcal{F}^n$  are dependent iff  $\det(V) \in \mathcal{G} \cup \{0_{\mathcal{F}}\}$ , where  $V$  is the matrix having  $v_1, \dots, v_n$  for its columns.

We define two types of special matrices:

**Definition 2.9.** An  $n \times n$  matrix  $P = (p_{i,j})$  is a **permutation matrix** if there exists  $\pi \in S_n$  such that

$$p_{i,j} = \begin{cases} 0_{\mathcal{F}}, & j \neq \pi(i) \\ 1_{\mathcal{F}}, & j = \pi(i) \end{cases}.$$

Since  $\forall \pi \in S_n \exists! \sigma \in S_n : \sigma = \pi^{-1}$  and  $1_{\mathcal{F}}$  is invertible, a permutation matrix is always invertible.

An  $n \times n$  matrix  $D = (d_{i,j})$  is a **diagonal matrix** if

$$\exists a_1, \dots, a_n \in \mathcal{F} : d_{i,j} = \begin{cases} 0_{\mathcal{F}}, & j \neq i \\ a_i, & j = i \end{cases},$$

which is invertible if and only if  $\det(D)$  is invertible (i.e.,  $a_i \in \mathcal{T}$ ,  $\forall i$ ).

*Remark 2.10.* (See [20, Proposition 3.9]) A tropical matrix  $A$  is invertible if and only if it is a product of a permutation matrix and an invertible diagonal matrix. These types of products are called **generalized permutation matrices**, that is  $(d_{i,j})$  such that

$$\exists a_1, \dots, a_n \in \mathcal{T}, \pi \in S_n : d_{i,j} = \begin{cases} 0_{\mathcal{F}}, & j \neq \pi(i) \\ a_i, & j = \pi(i) \end{cases}.$$

We define three types of tropical elementary matrices, corresponding to the three elementary matrix operations, obtained by applying one such operation to the identity matrix.

A **transposition matrix** is obtained from the identity matrix by switching two rows (resp. columns). This matrix is invertible:  $E_{i,j}^{-1} = E_{i,j}$ , and a product of transposition matrices yields a permutation matrix.

An **elementary diagonal multiplier** is obtained from the identity matrix where one row (resp. column) has been multiplied by an invertible scalar. This matrix is invertible:  $E_{\alpha, i^{th} row}^{-1} = E_{\alpha^{-1}, i^{th} row}$ , and a product of diagonal multipliers yields an invertible diagonal matrix.

A **Gaussian matrix** is defined to differ from the identity matrix by having a non-zero entry in a non-diagonal position. We denote as  $E_{i^{th} row + \alpha \cdot j^{th} row}$  the elementary Gaussian matrix adding row  $j$ , multiplied by  $\alpha$ , to row  $i$ . By Remark 2.10, this matrix is not invertible.

**Definition 2.11.** A nonsingular matrix  $A = (a_{i,j})$  is defined as **definite** if

$$\det(A) = 0 = a_{i,i}, \forall i.$$

**2.2.1. The supertropical approach.** Having established that algebraically  $\mathcal{G} \cup \{-\infty\}$  and  $\models_{gs}$  effectively take the role of singularity and equality over  $\mathcal{F}$ , we would like to extend additional definitions to the supertropical setting, using ghosts for zero.

A **quasi-zero** matrix  $Z_{\mathcal{G}}$  is a matrix equal to  $0_{\mathcal{F}}$  on the diagonal, and whose off-diagonal entries are ghost or  $0_{\mathcal{F}}$ .



A **diagonally dominant** matrix is a nonsingular matrix with a dominant permutation along the diagonal.

A **quasi diagonally dominant** matrix  $D_G$  is a diagonally dominant matrix  $A$  whose off-diagonal entries are ghost or  $0_{\mathcal{F}}$ .

A **quasi-identity** matrix  $I_G$  is a nonsingular, multiplicatively idempotent matrix equal to  $I + Z_G$ , where  $Z_G$  is a quasi-zero matrix.

Thus, every quasi-identity matrix  $I_G$  is quasi diagonally dominant. Using the tropical determinant, we attain the tropical analog for the well-known *adjoint*.

**Definition 2.12.** The  $r, c$ -**minor**  $A_{r,c}$  of a matrix  $A = (a_{i,j})$  is obtained by deleting row  $r$  and column  $c$  of  $A$ . The **adjoint matrix**  $\text{adj}(A)$  of  $A$  is defined as the matrix  $(a'_{i,j})$ , where  $a'_{i,j} = \det(A_{j,i})$ . When  $\det(A)$  is invertible, the matrix  $A^\nabla$  denotes

$$\frac{1}{\det(A)} \text{adj}(A).$$

Notice that  $\det(A_{j,i})$  may be obtained as the sum of all permutations in  $A$  passing through  $a_{j,i}$ , but with  $a_{j,i}$  deleted:

$$\det(A_{j,i}) = \sum_{\substack{\sigma \in S_n : \\ \sigma(j) = i}} a_{1,\sigma(1)} \cdots a_{j-1,\sigma(j-1)} a_{j+1,\sigma(j+1)} \cdots a_{n,\sigma(n)}.$$

When writing each permutation as the product of disjoint cycles,  $\det(A_{j,i})$  can be presented as:

$$\det(A_{j,i}) = \sum_{\substack{\sigma \in S_n : \\ \sigma(j) = i}} (a_{i,\sigma(i)} a_{\sigma(i),\sigma^2(i)} \cdots a_{\sigma^{-1}(j),j}) C_\sigma,$$

where  $C_\sigma$  is the product of the remaining cycles.

**Definition 2.13.** We say that  $A^\nabla$  is the **quasi-inverse** of  $A$  over  $\mathcal{F}$ , denoting

$$I_A = AA^\nabla \text{ and } I'_A = A^\nabla A,$$

where  $I_A, I'_A$  are quasi-identities (see [21, Theorem 2.8]).

These supertropical definitions provide a tropical version for two well-known algebraic properties, proved in Proposition 4.8. and Theorem 4.9. of [20].

**Proposition 2.14.**  $\text{adj}(AB) \models_{gs} \text{adj}(B) \text{adj}(A)$ .

As a result, one concludes from the fourth property of  $\models_{gs}$  (see Definition 1.1) and Theorem 2.6 that  $(AB)^\nabla \models_{gs} B^\nabla A^\nabla$ , when  $AB$  is nonsingular.

**Theorem 2.15.**

- (i)  $\det(A \cdot \text{adj}(A)) = \det(A)^n$ .
- (ii)  $\det(\text{adj}(A)) = \det(A)^{n-1}$ .

*Remark 2.16.* (see [28, Remark 2.18]) For a definite matrix  $A$  we have

$$A^\nabla = \frac{1}{\det(A)} \text{adj}(A) = \text{adj}(A),$$

which is also definite.

The following lemma has been proved in [28, Lemma 3.2], and states the connection between multiplicity of the determinant and the quasi-inverse matrix:

**Lemma 2.17.** *Let  $P$  be an invertible matrix and  $A$  be nonsingular.*

- (i)  $P^\nabla = P^{-1}$ .
- (ii)  $\det(PA) = \det(P) \det(A)$ .
- (iii)  $(PA)^\nabla = A^\nabla P^\nabla$ .
- (iv) *If  $A = P\bar{A}$ , where  $\bar{A}$  is the definite form of  $A$  with left normalizer  $P$ , then  $A^\nabla = \bar{A}^\nabla P^{-1}$  where  $\bar{A}^\nabla$  is definite, with right normalizer  $P^{-1}$ .*

### Matrix invariants

Let  $A \in M_n(\mathcal{F})$ . We continue the supertropical approach by defining  $v \in V$ , not all singular, such that  $\exists \lambda \in \mathcal{T} \cup \{0_{\mathcal{F}}\}$  where  $Av \models_{gs} \lambda v$ , to be a **supertropical eigenvector** of  $A$  with a **supertropical eigenvalue**  $\lambda$ , having an **eigenmatrix**  $A + \lambda I$ . The **eigenspace**  $V_\lambda$  is the set of eigenvectors with eigenvalue  $\lambda$ .

The **characteristic polynomial** of  $A$  (also called the maxpolynomial, cf.[8]) is defined to be

$$f_A(x) = \det(xI + A).$$

The tangible value of its roots are the eigenvalues of  $A$ , as shown in [20, Theorem 7.10]. Following to Definition 2.4, we may have *corner eigenvalues* and *non-corner eigenvalues*.

The coefficient of  $x^{n-k}$  in this polynomial is the sum of determinants of all  $k \times k$  **principal sub-matrices**, otherwise known as the trace of the  $k^{th}$  compound matrix of  $A$ . Thus, this coefficient, which we denote as  $\alpha_k$ , takes the dominant value among the permutations on all subsets of indices of size  $k$ :

$$\alpha_k = \sum_{\substack{I \subseteq [n] : \\ |I| = k}} \sum_{\sigma \in S_k} \prod_{i \in I} a_{i, \sigma(i)}.$$

When  $\alpha_k \in \mathcal{T}$ , we define the **index set of  $\alpha_k$** , denoted by  $\text{Ind}_k$ , a set  $I \subseteq [n]$  on which the dominant permutation defining  $\alpha_k$  is obtained.

Let  $f_A(x) = \sum_{i=0}^n \alpha_i x^{n-i}$  be the characteristic polynomial of  $A$ , with the essential polynomial

$$f_A^{es}(x) = \sum_k \alpha_{i_k} x^{n-i_k}.$$

Let  $\lambda$  be the corner eigenvalue obtained between the essential monomial  $\alpha_{i_{k-1}} x^{n-(i_{k-1})}$  and the subsequent essential monomial  $\alpha_{i_k} x^{n-i_k}$ . We denote  $I_\lambda = \text{Ind}_{i_k} \setminus \text{Ind}_{i_{k-1}}$ .

**Theorem 2.18.** *(The eigenvectors algorithm, see [21, Remark 5.3 and Theorem 5.6].) Let  $t \in I_\lambda$ . The tangible value of the  $t^{th}$ -column of  $\text{adj}(\lambda I + A)$  (see Notation 2.1), is a tropical eigenvector of  $A$  with respect to the eigenvalue  $\lambda$ .*

This algorithm will be demonstrated in §3.2.

The Supertropical Cayley-Hamilton Theorem has been proved in [20, Theorem 5.2], and is as follows:

**Theorem 2.19.** *Any matrix  $A$  satisfies its tangible characteristic polynomial  $f_A$ , in the sense that  $f_A(A)$  is ghost.*



One can find a combinatorial proof in [30] and a proof using the transfer principle in [2].

In analogy to the classical theory, we have

**Proposition 2.20.** ([20, Proposition 7.7]) *The roots of the polynomial  $f_A(x)$  are precisely the supertropical eigenvalues of  $A$ .*

*Remark 2.21.* Recall that a supertropical polynomial is  **$r$ -primary** if it has the unique supertropical root  $r$ . It is well-known that any tropical  $r$ -primary polynomial has the form  $(x + r)^m$  for some  $m \in \mathbb{N}$ , and any tropical essential polynomial  $f_A$  can be factored as a function to a product of primary polynomials, and thus of the form  $\prod_i g_i$  where  $g_i = (x + r_i)^{m_i}$ . The supertropical version of this is given in [19, Theorem 8.25 and Theorem 8.35].

Another classical property attained in this extended structure is:

**Proposition 2.22.** *If  $\lambda \in \mathcal{T} \cup \{0_{\mathcal{F}}\}$  is a supertropical eigenvalue of a matrix  $A \in M_n(\mathcal{F})$  with eigenvector  $v$ , then  $\lambda^i$  is a supertropical eigenvalue of  $A^i$ , for every  $i \in \mathbb{N}$ , with respect to the same eigenvector.*

**Theorem 2.23.** *Let  $A$  be a nonsingular matrix.*

(1) ([27, Theorem 3.6]) *For any  $m \in \mathbb{N}$  we have*

$$f_{A^m}(x^m) \models_{gs} (f_A(x))^m,$$

*implying that the  $m^{\text{th}}$ -root of every corner eigenvalue of  $A^m$  is a corner eigenvalue of  $A$ .*

(2) ([6, Theorem 4.1]) *For  $A^\nabla$ , the quasi-inverse of  $A$ , we have*

$$\det(A) f_{A^\nabla}(x) \models_{gs} x^n f_A(x^{-1}),$$

*implying that the inverse of every corner eigenvalue of  $A^\nabla$  is a corner eigenvalue of  $A$ .*

### 3. DEPENDENCE OF EIGENVECTORS

A well-known decomposition of  $F^n$ , where  $F$  is a field, is the decomposition to eigenspaces of a matrix  $A \in M_n(F)$ . In particular, this decomposition is obtained when the eigenvalues are distinct since, in the classical case, eigenspaces of distinct eigenvalues are linearly independent, which compose a basis for  $F^n$ . In the tropical case, considering that dependence occurs when a tropical linear combination ghost-surpasses  $\vec{0}_{\mathcal{F}}$ , such a property need not necessarily hold.

In the upcoming section we analyze the dependence between eigenvectors, using their definition according to the algorithm described in Theorem 2.18. We present special cases in which this undesired dependence is resolved.

**Definition 3.1.** The matrix  $A$  satisfies the **difference criterion** if the sets  $I_\lambda$ , such that  $\lambda$  is a corner root of  $f_A$ , are disjoint.

**3.1. Eigenspaces in lower dimensions.** In the following proposition, we verify independence of eigenvectors having distinct eigenvalues, for dimensions  $n = 2, 3$ .

**Proposition 3.2.** *Let  $A = (a_{i,j})$  be a nonsingular  $n \times n$  matrix, where  $n \in \{2, 3\}$ , with a tangible characteristic polynomial (coefficient-wise) and  $n$  distinct eigenvalues. Then the eigenvectors of  $A$  are tropically independent.*

*Proof.*

The  $2 \times 2$  case:

Let  $f_A(x) = x^2 + \text{tr}(A)x + \det(A)$  be the characteristic polynomial of  $A$ . If  $A$  has two distinct eigenvalues, then these must be  $\lambda_1 = \text{tr}(A)$  and  $\lambda_2 = \frac{\det(A)}{\text{tr}(A)}$ .

We must have  $\lambda_1 > \lambda_2$ , for otherwise either

$$f_A(\lambda_2) = \frac{\det(A)}{\text{tr}(A)} \left( \frac{\det(A)}{\text{tr}(A)} + \text{tr}(A)^\nu \right) = \left( \frac{\det(A)}{\text{tr}(A)} \right)^2 \in \mathcal{T},$$

or  $\lambda_1 = \lambda_2$ , which means the polynomial has one root with multiplicity 2.

Without loss of generality, we may assume that  $\text{tr}(A) = a_{1,1}$ . According to the algorithm, since  $I_{\lambda_1} = \{1\}$ ,  $\lambda_1$  has the eigenvector obtained by the tangible value of the *first* column of its eigenmatrix. Since  $I_{\lambda_2} = \{2\}$ ,  $\lambda_2$  has the eigenvector obtained by the tangible value of the *second* column of its eigenmatrix.

The determinant is either:

$$\det(A) = a_{1,1}a_{2,2}, \text{ where } a_{1,1} > a_{2,2} \text{ and } a_{1,1}a_{2,2} > a_{1,2}a_{2,1},$$

(and then the eigenvalues are  $a_{1,1}$  and  $a_{2,2}$ ), or

$$\det(A) = a_{1,2}a_{2,1}, \text{ where } a_{1,1}a_{2,2} < a_{1,2}a_{2,1},$$

(and then the eigenvalues are  $a_{1,1}$  and  $\frac{a_{1,2}a_{2,1}}{a_{1,1}}$ , satisfying  $a_{1,1} > \frac{a_{1,2}a_{2,1}}{a_{1,1}} > a_{2,2}$ ).

In both cases, the first column of  $\text{adj}(A + \lambda_1 I)$  is  $(a_{1,1}, a_{2,1})$  and the second column of  $\text{adj}(A + \lambda_2 I)$  is  $(a_{1,2}, a_{1,1})$ , which are tropically independent since  $a_{1,1}^2 > a_{1,2}a_{2,1}$ .

The  $3 \times 3$  case:

This case indicates key techniques for understanding and motivating the general proof on matrices satisfying the difference criterion in §3.3.1.

Let  $f_A(x) = x^3 + \text{tr}(A)x^2 + \alpha x + \det(A)$  be the characteristic polynomial of  $A$ , recalling that  $\alpha$  is the sum of the determinants of all of the principle  $2 \times 2$  sub-matrices. We assign  $\text{tr}(A)$  to be  $a_{1,1}$ , i.e.,

$$(3.1) \quad a_{1,1} > a_{t,t} \quad \forall t \neq 1.$$

For the determinant we have six permutations of  $S_3$ . In order to obtain three distinct eigenvalues, we must have

$$(3.2) \quad \lambda_1 = \text{tr}(A) > \lambda_2 = \frac{\alpha}{\text{tr}(A)} > \lambda_3 = \frac{\det(A)}{\alpha},$$

for otherwise  $\exists t, s : f_A(\lambda_t) \in \mathcal{T}$  or  $\lambda_t = \lambda_s$ . Thus

$$(3.3) \quad \lambda_1 \lambda_2 = \alpha \quad \text{and} \quad \lambda_1 \lambda_2 \lambda_3 = \det(A).$$

As a result,  $\text{Ind}_1 \subseteq \text{Ind}_2$ ; otherwise,  $a_{1,1}$  together with  $\alpha$  yields a permutation whose weight is dominated by  $\det(A)$ , and we get  $\lambda_1 = a_{1,1} < \frac{\det(A) \cdot a_{1,1}}{\alpha \cdot a_{1,1}} = \lambda_3$ , contrary to (3.2).

Therefore,

$$\begin{cases} I_{\lambda_1} = \{1\} \setminus \emptyset = \{1\} \\ I_{\lambda_2} = \{1, j\} \setminus \{1\} = \{j\}, \\ I_{\lambda_3} = \{1, j, k\} \setminus \{1, j\} = \{k\}, \end{cases}$$

where  $1, j, k$  are distinct. Without loss of generality, we may take  $j = 2$  and  $k = 3$ , and obtain the eigenmatrices:

$$A + \lambda_1 I = \begin{pmatrix} \lambda_1 & a_{1,2} & a_{1,3} \\ a_{2,1} & \lambda_1 & a_{2,3} \\ a_{3,1} & a_{3,2} & \lambda_1 \end{pmatrix}, \text{ since } \operatorname{tr}(A) = a_{1,1} > a_{t,t}, \quad \forall t \neq 1, \text{ by (3.1),}$$

$$A + \lambda_2 I = \begin{pmatrix} \lambda_1 & a_{1,2} & a_{1,3} \\ a_{2,1} & \lambda_2 & a_{2,3} \\ a_{3,1} & a_{3,2} & \lambda_2 \end{pmatrix}, \text{ since } \underbrace{\frac{\alpha}{\operatorname{tr}(A) \cdot a_{t,t}}}_{\geq 1} a_{t,t} \geq a_{t,t},$$

because  $\operatorname{tr}(A) \cdot a_{t,t}$  is a summand of  $\alpha$ ,  $\forall t \neq 1$ , and

$$A + \lambda_3 I = \begin{pmatrix} \lambda_1 & a_{1,2} & a_{1,3} \\ a_{2,1} & \beta & a_{2,3} \\ a_{3,1} & a_{3,2} & \lambda_3 \end{pmatrix}, \text{ since } \underbrace{\frac{\det(A)}{\alpha \cdot a_{3,3}}}_{\geq 1} a_{3,3} \geq a_{3,3},$$

where  $\beta = \max\{a_{2,2}, \lambda_3\}$ , since  $\alpha \cdot a_{3,3}$  is a summand in  $\det(A)$ .

Recalling the algorithm in Theorem 2.18, we let  $W$  be the matrix with the (tangible value of the) eigenvectors for its columns

$$W = \begin{pmatrix} \lambda_1^2 & a_{1,2}\lambda_2 + a_{1,3}a_{3,2} & a_{1,3}\beta + a_{1,2}a_{2,3} \\ a_{2,1}\lambda_1 + a_{2,3}a_{3,1} & \lambda_1\lambda_2 & a_{2,3}\lambda_1 + a_{2,1}a_{1,3} \\ a_{3,1}\lambda_1 + a_{3,2}a_{2,1} & a_{3,2}\lambda_1 + a_{3,1}a_{1,2} & \lambda_1\beta + a_{1,2}a_{2,1} \end{pmatrix}.$$

We get  $W_{3,3} = \lambda_1\lambda_2$ , since

if  $\alpha = \lambda_1\lambda_2 = a_{1,2}a_{2,1}$  then  $\lambda_1 a_{2,2} < \alpha$ ,  $\lambda_1\lambda_3 < \alpha \Rightarrow \lambda_1\beta + a_{1,2}a_{2,1} = \lambda_1\lambda_2$ , and

if  $\alpha = \lambda_1\lambda_2 = a_{1,1}a_{2,2}$  then  $\lambda_3 < \frac{a_{2,2}}{a_{1,1}}a_{1,1} = \lambda_2 \Rightarrow \beta = a_{2,2}$ ,  $\lambda_1 a_{2,2} + a_{1,2}a_{2,1} = \lambda_1\lambda_2$ .

Due to relations (3.1)-(3.3), all non-identity permutations in  $\det(W)$ :

$$\begin{cases} \lambda_1^4 a_{2,3} a_{3,2} = \lambda_1^2 \lambda_1 a_{1,1} a_{2,3} a_{3,2} \leq \lambda_1^2 \lambda_1 \lambda_1 \lambda_2 \lambda_3 & \text{and } \lambda_1^2 \lambda_2 \lambda_3 a_{1,3} a_{3,1} < \lambda_1^2 \lambda_1 \lambda_2 a_{1,3} a_{3,1}, \\ \lambda_1 \lambda_m \lambda_r a_{i,j} a_{j,l} a_{l,i} \leq \lambda_1^2 \lambda_2 (\lambda_1 \lambda_2 \lambda_3) & , i, j, l \text{ distinct,} \\ \lambda_m \lambda_r (a_{i,j} a_{j,i}) (a_{i,l} a_{l,i}) \leq \lambda_1^2 (a_{i,j} a_{j,i}) (a_{i,l} a_{l,i}) & , i, j, l \text{ distinct,} \\ \lambda_m (a_{i,j} a_{j,i}) (a_{i,l} a_{l,t} a_{t,i}) \leq \lambda_1 (\lambda_1 \lambda_2) (\lambda_1 \lambda_2 \lambda_3) & , j \neq i, t, l \text{ distinct,} \\ (a_{i,j} a_{j,i}) (a_{k,l} a_{l,k}) (a_{t,s} a_{s,t}) \leq (\lambda_1 \lambda_2)^3 & , i \neq j, k \neq l, t \neq s, \\ \text{and } (a_{i,j} a_{j,k} a_{k,i}) (a_{l,t} a_{t,s} a_{s,l}) \leq (\lambda_1 \lambda_2 \lambda_3)^2 & , i, j, k \text{ distinct, } s, t, l \text{ distinct,} \end{cases}$$

are strictly dominated by  $\lambda_1^2 (\lambda_1 \lambda_2) (\lambda_1 \lambda_2)$ .

□

We further study this property in the generalization proved in Theorem 3.4. The cases in Step 3 of its proof are demonstrated above.

**3.2. The pathology appears.** We follow Example 3.3, introduced in [21], to show how independence of eigenspaces might fail for dimensions higher than 3, due to the increased variety of indices. While applying the eigenvectors-algorithm, we utilize a supertropical analog of classical Gaussian elimination, treating the ghosts as “zero-elements”. This illustrative example will provide the motivation for Theorem 3.4, Conjecture 3.5 and Conjecture 3.6, generalizing the connection of the index sets to the dependence of the eigenvectors.

*Example 3.3.* Let

$$A = \begin{pmatrix} 10 & 10 & 9 & - \\ 9 & 1 & - & - \\ - & - & - & 9 \\ 9 & - & - & - \end{pmatrix}.$$

The characteristic polynomial of  $A$  is

$$f_A(x) = x^4 + 10x^3 + 19x^2 + 27x + 28,$$

obtained from the permutations (1), (1 2), (1 3 4), (1 3 4)(2), respectively. Therefore,

$$(3.4) \quad \begin{cases} I_{\lambda_1} = \{1\} \setminus \emptyset = \{1\}, \\ I_{\lambda_2} = \{1, 2\} \setminus \{1\} = \{2\}, \\ I_{\lambda_3} = \{1, 3, 4\} \setminus \{1, 2\} = \{3, 4\}, \\ I_{\lambda_4} = \{1, 2, 3, 4\} \setminus \{1, 3, 4\} = \{2\} \end{cases}$$

where  $\lambda_1 = 10$ ,  $\lambda_2 = 9$ ,  $\lambda_3 = 8$  and  $\lambda_4 = 1$ , are the eigenvalues of  $A$ . As we saw in §3.1, the overlap of the second and fourth sets cannot occur in lower dimensions.

The eigenmatrices and eigenvectors are as follows:

For  $\lambda_1$  :

$$A + 10I = \begin{pmatrix} 10^\nu & 10 & 9 & - \\ 9 & 10 & - & - \\ - & - & 10 & 9 \\ 9 & - & - & 10 \end{pmatrix},$$

and the tangible value of the first column of its adjoint is

$$v_1 = (30, 29, 28, 29) = 28(2, 1, 0, 1).$$

This can also be obtained when multiplying the eigenmatrix by

$$E_{4^{th} \text{ row}+1 \cdot 3^{rd} \text{ row}}^2 E_{4^{th} \text{ row}+1 \cdot 2^{nd} \text{ row}} E_{2^{nd} \text{ row}+1^{st} \text{ row}} E_{1,4}$$

on the left:

$$\begin{pmatrix} 9 & - & - & 10 \\ 9^\nu & 10 & - & 10 \\ - & - & 10 & 9 \\ 10^\nu & 10^\nu & 12^\nu & 11^\nu \end{pmatrix},$$

and solving the tropically linear system

$$\begin{cases} 9x + 10w \in \mathcal{G}, \\ 10y + 10w \in \mathcal{G}, \\ 10z + 9w \in \mathcal{G}, \end{cases}$$

which yields  $(11, 10, 9, 10) = 9(2, 1, 0, 1)$ , a multiple of  $v_1$ .

For  $\lambda_2$ :

$$A + 9I = \begin{pmatrix} 10 & 10 & 9 & - \\ 9 & 9 & - & - \\ - & - & 9 & 9 \\ 9 & - & - & 9 \end{pmatrix},$$

and the tangible value of the second column of its adjoint is

$$v_2 = (28, 28, 28, 28) = 28(0, 0, 0, 0).$$

This can also be obtained when multiplying the eigenmatrix by

$$E_{4^{th} \text{ row}+2.3^{rd} \text{ row}} E_{4^{th} \text{ row}+1.2^{nd} \text{ row}} E_{2^{nd} \text{ row}+1^{st} \text{ row}} E_{1,4}$$

on the left:

$$\begin{pmatrix} 9 & - & - & 9 \\ 9^\nu & 9 & - & 9 \\ - & - & 10 & 9 \\ 10^\nu & 10^\nu & 9^\nu & 9^\nu \end{pmatrix},$$

and solving the tropically linear system

$$\begin{cases} 9x + 9w \in \mathcal{G}, \\ 9y + 9w \in \mathcal{G}, \\ 9z + 9w \in \mathcal{G}, \end{cases}$$

which yields  $(0, 0, 0, 0)$ , a multiple of  $v_2$ .

For  $\lambda_3$ :

$$A + 8I = \begin{pmatrix} 10 & 10 & 9 & - \\ 9 & 8 & - & - \\ - & - & 8 & 9 \\ 9 & - & - & 8 \end{pmatrix},$$

and the tangible value of the third column of its adjoint is

$$v_3 = (25, 26, 27, 26) = 25(0, 1, 2, 1).$$

This can also be obtained when multiplying the eigenmatrix by

$$E_{4^{th} \text{ row}+1.3^{rd} \text{ row}} E_{4^{th} \text{ row}+2.2^{nd} \text{ row}} E_{2^{nd} \text{ row}+1^{st} \text{ row}} E_{1,4}$$

on the left:

$$\begin{pmatrix} 9 & - & - & 8 \\ 9^\nu & 8 & - & 8 \\ - & - & 8 & 9 \\ 11^\nu & 10^\nu & 9^\nu & 10^\nu \end{pmatrix},$$

and solving the tropically linear system

$$\begin{cases} 9x + 8w \in \mathcal{G}, \\ 8y + 8w \in \mathcal{G}, \\ 8z + 9w \in \mathcal{G}, \end{cases}$$

which yields  $(7, 8, 9, 8) = 7(0, 1, 2, 1)$ , a multiple of  $v_3$ .

For  $\lambda_4$

$$A + 1I = \begin{pmatrix} 10 & 10 & 9 & - \\ 9 & 1^\nu & - & - \\ - & - & 1 & 9 \\ 9 & - & - & 1 \end{pmatrix},$$

and the tangible value of the second column of its adjoint is

$$v_4 = (12, 27, 28, 20) = 12(0, 15, 16, 8).$$

This can also be obtained when multiplying the eigenmatrix by

$$E_{4^{th}row+(-1) \cdot 1^{st} row} E_{4^{th}row+2^{nd} row} E_{2^{nd}+(-1) \cdot 1^{st} row}$$

on the left:

$$\begin{pmatrix} 10 & 10 & 9 & - \\ 9^\nu & 9 & 8 & - \\ - & - & 1 & 9 \\ 9^\nu & 9^\nu & 8^\nu & 1^\nu \end{pmatrix},$$

and solving the tropically linear system

$$\begin{cases} 10x + 10y + 9z \in \mathcal{G}, \\ 9y + 8z \in \mathcal{G}, \\ 1z + 9w \in \mathcal{G}, \end{cases}$$

which yields  $(x, 8, 9, 1)$ , where  $x \leq 8$ .

From the fourth position of  $Av \models_{gs} \lambda v$ , we get  $9x \models_{gs} 2$  which implies  $x = -7$ . Thus the eigenvector is  $(-7, 8, 9, 1) = -7(0, 15, 16, 8)$ , a multiple of  $v_4$ .

Next, we examine the dependence of the eigenvectors, using the matrix  $W$  having these vectors for its columns:

$$W = \begin{pmatrix} 30 & 28 & 25 & 12 \\ 29 & 28 & 26 & 27 \\ 28 & 28 & 27 & 28 \\ 29 & 28 & 26 & 20 \end{pmatrix}.$$

The determinant of  $W$  is  $112^\nu$  and is obtained by the permutations  $(1)(2)(3\ 4)$  and  $(1)(2\ 4)(3)$ . One can see that the ghost part of the product is attained in the principal sub-matrix  $\{2, 3, 4\} \times \{2, 3, 4\}$ , where the pathology of the index sets occurs. We rewrite  $W$  using the eigenvalues and the entries of  $A = (a_{i,j})$ , in order to understand this dependence:

$$W = \begin{pmatrix} \lambda_1^3 & a_{1,2}\lambda_2^2 & \lambda_3^2 a_{1,3} & \lambda_4^2 a_{1,2} \\ \lambda_1^2 a_{2,1} & \lambda_1 \lambda_2^2 & \lambda_3 a_{2,1} a_{1,3} & a_{1,3} a_{3,4} a_{4,1} \\ \lambda_1 a_{3,4} a_{4,1} & a_{3,4} a_{4,1} a_{1,2} & \lambda_3 a_{1,2} a_{2,1} & a_{3,4} a_{4,1} a_{1,2} \\ \lambda_1^2 a_{4,1} & \lambda_2 a_{4,1} a_{1,2} & \lambda_3 a_{4,1} a_{1,3} & \lambda_4 a_{4,1} a_{1,2} \end{pmatrix}.$$

The determinant is attained by

$$\lambda_1^3 (\lambda_1 \lambda_2^2) (a_{3,4} a_{4,1} a_{1,2}) (\lambda_3 a_{4,1} a_{1,3}) \text{ and } \lambda_1^3 (a_{1,3} a_{3,4} a_{4,1}) (\lambda_3 a_{1,2} a_{2,1}) (\lambda_2 a_{4,1} a_{1,2}),$$

where all elements are identical, and  $\lambda_1 \lambda_2 = a_{1,2} a_{2,1}$ . That is, the ghost determinant is not an occasional outcome of repeated values (such as 9, 10 in the entries of  $A$ ), or some relations between coefficients. The singularity which we encounter is systematic:

$$\lambda_1^3 \underbrace{(\lambda_1 \lambda_2^2)}_{\alpha \lambda_2} (a_{4,1} a_{1,2} a_{3,4}) (a_{4,1} a_{1,3} \lambda_3) = \lambda_1^3 [\lambda_2 a_{4,1} a_{1,2}] [a_{3,4} a_{4,1} a_{1,3}] [a_{1,2} a_{2,1} \lambda_3].$$

**3.3. Resolving the pathology.** In this section we offer sufficient conditions for independence, and present two conjectures on the eigenvectors of the quasi-inverse of a matrix.

**3.3.1. The resolution by means of disjoint index sets.** The intersection of the  $\{I_\lambda\}$  causes the eigenvector dependency seen in the previous section. This pathology will be resolved in the following theorem using disjoint  $\{I_\lambda\}$ , in which we show that it is a Zariski-closed condition.

**Theorem 3.4.** *Let  $A = (a_{i,j})$  be a nonsingular  $n \times n$  matrix, with tangible characteristic polynomial (coefficient-wise) and  $n$  distinct eigenvalues. If  $A$  satisfies the difference criterion, then the eigenvectors of  $A$  are tropically independent.*

*Proof.* Let  $f_A(x) = \sum_{i=0}^n \alpha_i x^{n-i} \in \mathcal{T}[x]$  be the characteristic polynomial of  $A$ , which means  $\alpha_0 = 0$ ,  $\alpha_1 = \text{tr}(A)$ ,  $\alpha_n = \det(A)$ . Without loss of generality,  $\text{tr}(A) = a_{1,1}$ , i.e.,

$$(3.5) \quad \lambda_1 = a_{1,1} > a_{t,t} \quad \forall t \neq 1.$$

In order to get  $n$  distinct eigenvalues, we must have  $f_A(x) = f_A^{es}(x)$ , or equivalently

$$(3.6) \quad \lambda_1 = \text{tr}(A) > \lambda_2 = \frac{\alpha_2}{\text{tr}(A)} > \lambda_3 = \frac{\alpha_3}{\alpha_2} > \dots > \lambda_{n-1} = \frac{\alpha_{n-1}}{\alpha_{n-2}} > \lambda_n = \frac{\det(A)}{\alpha_{n-1}},$$

where  $\{\lambda_l\}_{l \in [n]}$  are the corner-roots of  $f_A$ . Otherwise,  $\exists t, s$  such that  $f_A(\lambda_t) \in \mathcal{T}$  or  $\lambda_t = \lambda_s$ , contrary to hypothesis. In particular,  $\text{tr}(A) = \lambda_1$  and  $\text{Ind}_1 = \{1\}$ .

We need to show that  $\det(W) \in \mathcal{T}$ , where  $W$  is the matrix of eigenvectors. This is achieved in three steps:

- (1) For every  $k \in [n]$ ,  $\text{Ind}_k \subseteq \text{Ind}_{k+1} \quad \forall k$ , and therefore  $I_{\lambda_k} = \{k\}$ .
- (2) For every  $k \in [n]$ ,  $W_{k,k} = \lambda_1 \cdots \lambda_{k-1} \lambda_k^{n-k}$ .
- (3) Finally,  $\det(W) = \prod_{k \in [n]} W_{k,k} \in \mathcal{T}$ , as desired.

**Step (1).** A straightforward application of (3.6) yields

$$(3.7) \quad \lambda_1 \cdots \lambda_k = \alpha_k \in \mathcal{T} \text{ and } \{1\} = \text{Ind}_1 \subseteq \text{Ind}_k, \quad \forall k \geq 1.$$

Otherwise,  $a_{1,1} \cdot \alpha_{k-1}$  would yield a permutation on  $k$  indices, dominated by  $\alpha_k$ :

$$\lambda_1 = a_{1,1} \leq \frac{\overbrace{\alpha_k}^{\geq 1_R}}{\underbrace{\alpha_{k-1} \cdot a_{1,1}}_{\geq 1_R}} \cdot a_{1,1} = \lambda_k, \quad \text{contradicting (3.6).}$$

Let  $\text{Ind}_k = \{1, j_2, \dots, j_k\}$ . Since  $\text{Ind}_0 = \emptyset$ , there exists  $i \leq k : j_s \in I_{\lambda_i}, \forall s \in \{2, \dots, k\}$ . Assume that  $\text{Ind}_{l-1} \subseteq \text{Ind}_l$  holds through  $l = k$ , and then fails for  $k+1$ . That is,

$$\forall l \leq k \quad \text{Ind}_{l-1} \subseteq \text{Ind}_l \quad \text{and} \quad \exists s \in \{2, \dots, k\} : j_s \notin \text{Ind}_{k+1}.$$



However, since  $\text{Ind}_n = [n]$ ,  $j_s \in \text{Ind}_n$ . We define  $t$  to be the minimal index  $k < t < n$  such that  $j_s \in \text{Ind}_t$  but  $j_s \notin \text{Ind}_{t-1}$ . That is  $j_s \in I_{\lambda_t} \cap I_{\lambda_i}$  for some  $i, t : i < k < t$ , contradicting the difference criterion. Therefore  $\text{Ind}_k \subseteq \text{Ind}_{k+1}$ ,  $\forall k$ .

**Step (2).** Up to some permutation, we may require w.l.g. that  $j_k = k$ ,  $\forall k \in [n]$ . That is,  $I_{\lambda_k} = \{k\}$   $\forall k \in [n]$ ,

$$(3.8) \quad a_{1,1} > \lambda_k = \underbrace{\frac{\alpha_k}{\alpha_{k-1} \cdot a_{t,t}}}_{\geq 1_R} \cdot a_{t,t} \geq a_{t,t}, \quad \forall t \geq k, \quad \forall k > 1,$$

with equality only when  $k = t$ , and

$$(3.9) \quad \beta_{k,t} = \max\{\lambda_k, a_{t,t}\}, \quad \forall t < k.$$

Thus, the entries of the  $k^{\text{th}}$  eigenmatrix  $A + \lambda_k I = (b_{i,j}^{(k)})$  are given by

$$(3.10) \quad b_{i,j}^{(k)} = \begin{cases} \lambda_1 & , i = j = 1 \\ \beta_{k,i} & , 1 < i = j < k \\ \lambda_k & , i = j \geq k \\ a_{i,j} & , i \neq j \end{cases}.$$

(For example, for  $k = 2$  and  $k = 3$  we get

$$\begin{pmatrix} \lambda_1 & & a_{i,j} : i < j \\ & \lambda_2 & \\ & & \ddots \\ a_{i,j} : i > j & & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda_1 & & a_{i,j} : i < j \\ & \beta_{3,2} & \\ & & \lambda_3 \\ & & & \ddots \\ a_{i,j} : i > j & & & \lambda_3 \end{pmatrix},$$

respectively, where  $a_{i,j}$  indicates that the off-diagonal entries are identical to those of  $A$ .)

Let  $\text{adj}(A) = (a'_{i,j})$ ,  $W = (w_{i,j})$  be the matrix with the (tangible value of the) eigenvectors for its columns, and notice that  $w_{k,k} = \text{adj}(A + \lambda_k I)_{k,k}$ .

- On one hand, by (3.10)  $(A + \lambda_k I)_{k,k} = \lambda_k$ . By [21, Theorem 2.8],

$$(3.11) \quad ((A + \lambda_k I) \text{adj}(A + \lambda_k I))_{k,k} = \det(A + \lambda_k I) = f_A(\lambda_k) \in \mathcal{G},$$

where  $f_A(\lambda_k) = \alpha_k \lambda_k^{n-k} + \alpha_{k-1} \lambda_k^{n-k+1} = (\lambda_1 \cdots \lambda_{k-1} \lambda_k^{n-k+1})^\nu$ , as  $\lambda_k$  is the  $k^{\text{th}}$  corner root of the polynomial of distinct coefficients  $f_A$ . Since every summand in (3.11) is dominated by this expression, we get

$$\text{adj}(A + \lambda_k I)_{k,k} = w_{k,k} \leq \lambda_1 \cdots \lambda_{k-1} \lambda_k^{n-k}.$$

- On the other hand,  $\lambda_k^{n-k} \det(M)$  is a summand in  $\text{adj}(A + \lambda_k I)_{k,k}$ , where  $M$  is the  $(k-1) \times (k-1)$ -principal sub-matrix of  $A + \lambda_k I$ , obtained by rows and columns  $[k-1]$ . Since  $A \leq A + \lambda_k I$  entry-wise (and in particular for  $M$  and its corresponding principal sub-matrix in  $A$ ), we get  $\det(M) \geq_\nu \alpha_{k-1} = \lambda_1 \cdots \lambda_{k-1}$ . Thus,  $\text{adj}(A + \lambda_k I)_{k,k} = w_{k,k} \geq \lambda_1 \cdots \lambda_{k-1} \lambda_k^{n-k}$ .

As a result,

$$(3.12) \quad w_{k,k} = \lambda_1 \cdots \lambda_{k-1} \lambda_k^{n-k}, \quad \forall k \in [n].$$

**Step (3).** Notice that

$$(3.13) \quad \prod_{k \in [n]} w_{k,k} = \prod_{k \in [n]} \lambda_1 \cdots \lambda_{k-1} \lambda_k^{n-k} = \prod_{k \in [n-1]} \lambda_1 \cdots \lambda_{k-1} \lambda_k^{n-k+1}.$$

We claim that any other permutation in  $W$  is strictly dominated by the term in (3.13).

Let  $X = (x_{i,j})$  be an  $n \times n$  matrix. For  $\pi \in S_n$  denote  $X_\pi = \prod_{i \in [n]} x_{i,\pi(i)}$ , and its cycles are referred to as  $X$ -cycles. An  $X$ -cycle of length  $d$  is said to be an  $X^{(d)}$ -cycle. Using (3.10), we denote by  $W_\pi^{(\lambda)}$  the product of eigenvalues of  $A$  in  $W_\pi$ . For example  $W_{\text{Id}}^{(\lambda)} = W_{\text{Id}}$ .

For every  $\pi \neq \text{Id}$ ,  $W_\pi = W_\pi^{(\lambda)} \cdot C$ , where  $C$  is a product of  $A$ -cycles, and  $W_\pi^{(\lambda)}$  is a product in  $W_{\text{Id}}$ . A cycle  $c \in C$  is an  $X^{(d)}$ -cycle for some  $d \in [n]$ , and is dominated by  $\lambda_1 \cdots \lambda_d$ , which is strictly dominated by  $\lambda_1 \cdots \lambda_{d-2} \lambda_{d-1}^2 < \lambda_1 \cdots \lambda_{d-3} \lambda_{d-2}^3 < \dots < \lambda_1^d$ . Therefore,  $W_\pi \leq W_{\text{Id}}$ , and we show strict dominance. The product  $W_\pi = W_\pi^{(\lambda)} \cdot C$  satisfies at least one of the following cases:

- $C$  includes an  $A^{(n)}$ -cycle, dominated by  $\lambda_1 \cdots \lambda_n$ , which is strictly dominated by  $\lambda_1 \cdots \lambda_{n-2} \lambda_{n-1}^2$  in (3.13).
- $C$  includes two different  $A^{(d)}$ -cycles, at least one is strictly dominated by  $\lambda_1 \cdots \lambda_d$  in (3.13).
- $C$  includes an  $A^{(d)}$ -cycle which does not act on some index of  $[d]$ , making it strictly dominated by  $\lambda_1 \cdots \lambda_d$  in (3.13).
- $C = \prod c$ , s.t.  $c$  is an  $A^{(d_c)}$ -cycle on indices  $[d_c]$ . Then,

$$W_\pi^{(\lambda)} = \prod_{j \in J \subseteq [n]} \lambda_j \Rightarrow \exists j \in J : j \neq 1,$$

whereby  $W_\pi^{(\lambda)}$  is strictly dominated by  $\lambda_1^m$  in (3.13), for  $m = |J|$ .

Since at least one term is strictly dominated, and the rest are dominated, the assertion follows.  $\square$

**3.3.2. The resolution by means of the quasi-inverse.** In view of the results in [6], [28] and [29], one can conclude that quasi-inverse matrices play an important role in formulating properties of matrices. These studies lead us to the following two conjectures, based on a further examination of Example 3.3.

**Conjecture 3.5.** *Let  $A$  be a nonsingular matrix with  $n$  distinct eigenvalues. If the eigenvectors of  $A$  are dependent, then*

- (1) *Recalling Theorem 2.23,  $\det(A) f_{A^\nabla}(x)$  strictly ghost-surpasses  $x^n f_A(x^{-1})$ .*
- (2) *The matrix  $A^\nabla$  has fewer distinct eigenvalues than  $A$ , when  $f_{A^\nabla} \neq f_{A^\nabla}^{\text{es}}$ .*
- (3) *Moreover, the eigenvectors of  $A^\nabla$  are independent.*

**Conjecture 3.6.** *Let  $A$  be a nonsingular matrix. If  $A^\nabla$  has  $n$  distinct eigenvalues, then their corresponding eigenvectors are independent.*

Let us consider Conjecture 3.5 in the case of Example 3.3. We recall Theorem 2.15 and Lemma 2.17, to conclude that  $\det(\text{adj}(A)) = \det(A)^{n-1}$  is attained solely by the permutation  $\sigma^{-1}$ , where  $\det(A)$  is attained solely by  $\sigma$ .

Let  $A$  be as in Example 3.3. As a result

$$\text{adj}(A) = \begin{pmatrix} - & - & - & 19 \\ - & 27 & - & 27 \\ 19 & 28 & - & 28 \\ - & - & 19 & - \end{pmatrix},$$

$$f_{\text{adj}(A)}(x) = x^4 + 27x^3 + 47x^2 + 74^\nu x + 84,$$

and  $\text{Ind}_1 = \{2\}$ ,  $\text{Ind}_2 = \{3, 4\}$ ,  $\text{Ind}_3 = \{2, 4, 3\} = \{2, 3, 4\}$ ,  $\text{Ind}_4 = \{3, 1, 4, 2\}$ .

(Indeed  $\det(\text{adj}(A)) = \det(A)^{4-1}$ , and  $f_{A^\nabla}$  is obtained by coefficients  $\frac{\alpha_k}{\det(A)^k}$ .)

It is easy to see that

$$\text{Ind}_1 \setminus \emptyset = \{2\}, \text{Ind}_2 \setminus \text{Ind}_1 = \{3, 4\}, \text{Ind}_3 \setminus \text{Ind}_2 = \{2\}, \text{Ind}_4 \setminus \text{Ind}_3 = \{1\}$$

are not disjoint. However, calculating the eigenvalues of  $\text{adj}(A)$  reveals these are not the sets  $I_{\lambda_k}$ . That is,  $f_{\text{adj}(A)}^{es}(x) = x^4 + 27x^3 + 74^\nu x + 84$ , and the dependence in the principal sub-matrix of  $\{2, 3, 4\}$  (identical to the minor causing dependence in  $W$ ),

$$\begin{aligned} \lambda_2(\lambda_1 \lambda_2)(a_{4,1}a_{1,2}a_{3,4})(a_{4,1}a_{1,3})\lambda_3 &= \lambda_2(a_{4,1}a_{1,2})(a_{3,4}a_{4,1}a_{1,3})(a_{1,2}a_{2,1})\lambda_3 \Rightarrow \\ (\lambda_1 \lambda_2)(a_{4,1}a_{1,2}a_{3,4})(a_{4,1}a_{1,3})\frac{a_{4,1}a_{1,3}a_{3,4}}{a_{1,1}} &= (a_{4,1}a_{1,2})(a_{3,4}a_{4,1}a_{1,3})(a_{1,2}a_{2,1})\frac{a_{4,1}a_{1,3}a_{3,4}}{a_{1,1}}, \end{aligned}$$

increases the coefficient of  $x$ , causing  $47x^2$  to be inessential. As a result,

$$I_{\lambda_1} = \{2\}, I_{\lambda_{2,3}} = \{3, 4\}, I_{\lambda_4} = \{1\},$$

where  $\lambda_1 = 27$ ,  $\lambda_{2,3} = 23.5$  (with multiplicity 2), and  $\lambda_4 = 10$ . As the conjecture predicted, the eigenvectors

$$v_1 = (66, 81, 82, 74) = 66(0, 15, 16, 8), \quad v_4 = (74, 65, 55, 65) = 55(19, 10, 0, 10),$$

$$\text{and } v_{2,3} = 65^{-1} \underbrace{(65, 69.5, 74, 69.5)}_{\text{from the third column}} = (0, 4.5, 9, 4.5) = 69.5^{-1} \underbrace{(69.5, 74, 78.5, 74)}_{\text{from the fourth column}}$$

are independent.

**3.3.3. The resolution by means of generalized eigenspaces.** **Eigenspaces** are studied in [21] and are defined in [22] to be spanned by supertropical eigenvectors. Let  $V = F^n$ .

**Definition 3.7.** A tangible vector  $v \in V$  is a **generalized supertropical eigenvector** of  $A$ , with **generalized supertropical eigenvalue**  $\lambda \in \mathcal{T}$ , if  $(A + \lambda I)^m v$  is ghost for some  $m \in \mathbb{N}$ . If  $A^m v$  is itself ghost for some  $m$ , we call the generalized eigenvector  $v$  **degenerate**.

The minimal such  $m$  is called the **multiplicity** of the eigenvalue (and also of the eigenvector).

The **generalized supertropical eigenspace**  $V_\lambda$  with **generalized supertropical eigenvalue**  $\lambda \in \mathcal{T}$  is the set of generalized supertropical eigenvectors with generalized supertropical eigenvalue  $\lambda$ .

Note that if  $v$  is a degenerate eigenvector, then it belongs to  $V_\lambda$  for all sufficiently small  $\lambda$ .

**Lemma 3.8.**  $V_\lambda$  is indeed a supertropical subspace of  $V$ .

*Proof.* Let  $v, u \in V_\lambda$ . Thus  $\exists m, t : (A + \lambda I)^m v \models_{gs} 0_{\mathcal{F}}$  and  $(A + \lambda I)^t u \models_{gs} 0_{\mathcal{F}}$ , and therefore for any  $a \in \mathcal{F}$

$$(A + \lambda I)^{(m+t)}(v + au) = (A + \lambda I)^t(A + \lambda I)^m v + a(A + \lambda I)^m(A + \lambda I)^t u \models_{gs} 0_{\mathcal{F}}.$$

□

*Remark 3.9.* We have the following hierarchy:

$Av \models_{gs} \lambda v$ , implies  $A^m v \models_{gs} \lambda^m v$ , implies  $A^m v + \lambda^m v \models_{gs} 0_{\mathcal{F}}$ , implies  $(A + \lambda I)^m v \models_{gs} 0_{\mathcal{F}}$ .

This approach gives some insight into the difference criterion. For the remainder of this paper we use the well-known digraph of a matrix, whose vertices are the indices  $\{1, \dots, n\}$  and whose edges correspond to the nonzero entries  $a_{i,j}$  of the matrix. Any permutation  $\pi$  corresponds to some cycle of length  $n$  which can be decomposed into disjoint simple cycles, and the contribution of the permutation to the determinant is the product of their weights. For any cycle of length  $k$  and weight  $\mu$ , its  $k$ -th power lies on the diagonal with all of the entries equal to  $\mu$  (so that its weight is  $\mu^k$ ). Thus, the corresponding part of the diagonal of  $A^k$  (and all subsequent powers) dominates all  $k$ -th powers cycles of length  $k$ , and in particular this is the case for  $A^m = (A^k)^{m/k}$ , for any multiple  $m$  of  $n!$ .

The diagonal is a dominant permutation of  $A^m$ .

**Lemma 3.10.** *The difference criterion is satisfied for  $A$  iff the diagonal entries of  $A^m$  are distinct, whenever  $n!$  divides  $m$ .*

*Proof.* ( $\Rightarrow$ ) The diagonal is a dominant permutation of  $A^m$ . The difference criterion implies that all of these diagonal entries are distinct.

( $\Leftarrow$ ) Suppose that in  $A^m$  some index  $i$  appears in both  $I_k$  and  $I_{k'}$  for  $k < k'$ , where  $k$  is taken minimal such. Then all the previous  $I_j$  are disjoint, so, rearranging the diagonal entries, we may assume that  $i$  appears in the  $|I_1| + \dots + |I_{k-1}| + \alpha$  position in the diagonal for some  $1 \leq \alpha \leq |I_k|$ . But  $i$  must also appear in the analogous position arising from  $I_{k'}$ , for some  $k' > k$ , so  $A^m$  has a double diagonal entry. □

**Lemma 3.11.** *If  $A$  is nonsingular and diagonally dominant, then the diagonal of  $A$  is tangible.*

*Proof.* The determinant is the product of the diagonal entries, so each is tangible. □

In view of Remark 2.21, we can refine the generalized supertropical eigenspaces  $V_\lambda$ . Write  $f_A = \prod_i g_i$  where  $g_i = (x + \lambda_i)^{t_i}$ , with the  $\lambda_i$  distinct, and let  $\tilde{f}_i = \prod_{j \neq i} g_j$ . (Thus,  $f_A = g_i \tilde{f}_i$ .) Suppose  $v \in \tilde{f}_i(A)V_\lambda$ . Then  $g_i(A)v \in f_A(A)V_\lambda$  is ghost, implying  $v \in V_\lambda$ . Thus, we can define the subspace

$$V'_{\lambda_i} = \left( \prod_{j \neq i} g_j(A) \right) V,$$

which is a generalized supertropical eigenspace with respect to  $\lambda_i$ .

**Definition 3.12.** A matrix  $A$  is **strongly nonsingular** if  $A^m$  is nonsingular for all  $m$ .

**Lemma 3.13.** *A strongly nonsingular matrix  $A$  has no nonzero degenerate generalized eigenvectors.*

*Proof.* Take  $m$  large enough (say  $n!$ ) such that  $A^m$  is dominated by the diagonal. Write  $A^m = (a_{i,j})$  and  $v = (v_1, \dots, v_n)$ . Then we have a contradiction to  $A^m v \in (\mathcal{G} \cup \{0_{\mathcal{F}}\})^n$  unless for each  $i$  there is  $i' = f(i)$  such that  $a_{i,i'} v_{i'} \geq a_{i,i} v_i$ . Write  $f^1 = f$  and  $f^k = f(f^{k-1})$ , and  $a_k = a_{f^k(i), f^{k-1}(i)}$ . Then  $f^k(i) = f^{k+t}(i)$  for  $t \leq n$ , and  $a_{k+t} \dots a_t \geq 1$ , contradicting  $A^{n!}$  nonsingular (since the dominant path is on the diagonal).  $\square$

**Theorem 3.14.** *If  $A$  is strongly nonsingular, then the  $V'_{\lambda_i}$  are independent*

*Proof.* We can replace  $A$  by  $A^{n!}$  and assume that  $A$  is diagonally dominant and that  $V'_{\lambda_i}$  are eigenspaces of  $A$ . We use the notation following Lemma 3.11.

We assume on the contrary that we have a ghost dependence, i.e.,  $\sum_{i \in [u]} \gamma_i \tilde{f}_i(A) v_i$  ghost for tangible  $\gamma_i$ , and aim for a contradiction. Since  $A$  is strongly nonsingular, the  $g_j$  act like scalar multiplication by  $\lambda_j$ , in view of Lemma 3.13, and, furthermore,  $\lambda_u^{t_u}$  dominates all  $\lambda_u^{t_u-j} \beta^j$ , for all  $\beta < \lambda_u$ . Hence, when  $x$  is to be specialized to these  $\beta$ ,  $\lambda_u^{t_u}$  dominates  $\sum_j \lambda_u^{t_u-j} x_u^j = g_u$ , and thus, by the argument of Lemma 3.13, some component of  $\gamma_u \lambda_u^{t_u} \tilde{f}_u(A) v_u$  is dominant in  $\gamma_u g_u(A) \tilde{f}_u(A) v_u$ , a ghost. Therefore some power of  $A$  ghost annihilates  $\tilde{f}_u(A) v_u = (\prod_{u' \neq u} \lambda_{u'}) v_u$ , contradicting  $A$  being strongly nonsingular.  $\square$

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